Equivalence of the Calogero-Sutherland Model to Free Harmonic Oscillators

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Abstract

A similarity transformation is constructed through which a system of particles interacting with inverse-square two-body and harmonic potentials in one dimension, can be mapped identically, to a set of free harmonic oscillators. This equivalence provides a straightforward method to find the complete set of eigenfunctions, the exact constants of motion and a *linear* $W_{1+\infty}$ algebra associated with this model. It is also demonstrated that a large class of models with long-range interactions, both in one and higher dimensions can be made equivalent to decoupled oscillators.

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In recent times, the one-dimensional system of identical particles having pair-wise inverse-square and harmonic interactions [1], known in the literature as the Calogero-Sutherland (CS) model, has generated wide interest. This exactly solvable model and its generalizations to the periodic case [2] and the spin systems [3], have been found relevant for the description of various physical phenomena such as the universal conductance fluctuations in mesoscopic systems [4], quantum Hall effect [5], wave propagation in stratified fields [6], random matrix theory [2,7], fractional statistics [8], two-dimensional gravity [9] and gauge theories [10].

Since its inception, this remarkable many-body system, with long-range interactions has been studied quite extensively in the literature for a better understanding of the origin of solvability and the underlying symmetries [11-13]. The energy eigenvalues and the level degeneracies of the CS model match identically with those of harmonic oscillators, apart from a coupling dependent shift of the ground-state energy. This structure of the spectrum has led Calogero to suggest the interesting possibility of a map between the CS model and decoupled oscillators [14].

In this paper, we provide a similarity transformation which realizes this correspondence by exactly mapping the CS system of interacting particles, to a set of free harmonic oscillators. This equivalence provides an elegant and straightforward method to construct the complete eigenstates of the CS model [15], including the degenerate ones, starting from the symmetrized form of the eigenstates of the harmonic oscillators and reveals the existence of a *linear* $W_{1+\infty}$ algebra as the infinite dimensional symmetry associated with the CS system. It also allows one to determine the N linearly independent, mutually commuting constants of motion and demonstrate that, a large class of CS type models in one and higher dimensions can also be solved in an analogous manner.

The N particle CS Hamiltonian is given by (in the units $\hbar = \omega = m = 1$)

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{i=1}^{N} x_i^2 + \frac{1}{2}g^2 \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{(x_i - x_j)^2} \qquad , \tag{1}$$

here, $g^2 > -\frac{1}{4}$ is the coupling constant. We work in a sector of the configuration space corresponding to a definite ordering of the particle coordinates: $x_1 \leq x_2 \leq \cdots \leq x_N$. This

is possible in one dimension since the particles can not overtake each other in the presence of the repulsive interaction. The resulting wave function is then analytically continued to other sectors of the configuration space, in order to obtain the actual eigenfunction.

The correlated ground-state of H is known to be of the form $\psi_0 = ZG$, where $Z \equiv \prod_{i<j}^{N}[|x_i - x_j|^\alpha (x_i - x_j)^\delta]$ and $G \equiv \exp\{-\frac{1}{2}\sum_i x_i^2\}$. The following similarity transformation then yields

$$\tilde{H} \equiv \psi_0^{-1} H \psi_0 = \sum_i x_i \frac{\partial}{\partial x_i} + E_0 - \hat{A} \qquad , \tag{2}$$

where, $E_0 = \frac{1}{2}N + \frac{1}{2}N(N-1)(\alpha+\delta)$ is the ground-state energy and $\hat{A} \equiv [\frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2} + (\alpha + \delta)\sum_{i\neq j}\frac{1}{(x_i - x_j)}\frac{\partial}{\partial x_i}]$. The eigenfunctions of $\tilde{H}$ must be totally symmetric with respect to the exchange of any two particle coordinates; the bosonic or the fermionic nature of the wave function being contained in the Jastrow factor Z. One can also show that,

$$\left[\tilde{H}, \exp\{-\hat{A}/2\}\right] = \left[\sum_i x_i \frac{\partial}{\partial x_i}, \exp\{-\hat{A}/2\}\right] = \hat{A}\exp\{-\hat{A}/2\} \qquad . \tag{3}$$

Making use of (3) in (2), it can be easily verified that, the operator $\hat{T} \equiv ZG\exp\{-\hat{A}/2\}$ diagonalizes the original Hamiltonian H:

$$\hat{T}^{-1} H \hat{T} = \sum_i x_i \frac{\partial}{\partial x_i} + E_0 \qquad . \tag{4}$$

The above equation is also obtained by Sogo [16]. The following similarity transformation on (4) achieves the correspondence of the CS model with the decoupled oscillators;

$$G\hat{E}\hat{T}^{-1} H \hat{T}\hat{E}^{-1} G^{-1} = -\frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_i x_i^2 + (E_0 - \frac{1}{2}N) \qquad , \tag{5}$$

where, $\hat{E} = \exp\{-\frac{1}{4}\sum_i \frac{\partial^2}{\partial x_i^2}\}$. As anticipated by Calogero and indicated by the structure of the eigenspectrum, only the ground-state energy depends on the coupling constant; the rest of the Hamiltonian describes N free oscillators. Hence, it follows that, the excited energy levels and the degeneracy structure of both the systems are identical, a fact known since the original solution of the interacting model [1].

From (4), one can define the creation and annihilation operators of CS model as $a_i^+ = \hat{T} x_i \hat{T}^{-1}$ and $a_i^- = \hat{T}\partial_i \hat{T}^{-1}$: $[a_i^- \ , \ a_j^+] = \delta_{ij}$ and the CS Hamiltonian becomes

$$H = \sum_i H_i = \frac{1}{2}\sum_i \{a_i^- \ , \ a_i^+\} \qquad , \tag{6}$$

where, $H_i \equiv \frac{1}{2}\{a_i^- \ , \ a_i^+\} + \frac{1}{2}[\alpha(N-1)]$ and $[H_i \ , \ a_i^-(a_i^+)] = -a_i^-(a_i^+)$.

One can also define $<< 0|S_n(\{a_i^-\}) =<< n|$ and $S_n(\{a_i^+\})|0>=|n>$ as the bra and ket vectors; S_n is a symmetric and homogeneous function of degree n and $<< 0|a_i^+ = a_i^-|0>=0$. Since the oscillators are decoupled, the inner product between these bra and ket vectors proves that any ket $|n>$, with a given partition of n, is orthogonal to all the bra vectors, with different n and also to those with different partitions of the same n.

For the construction of the eigenfunctions of CS model, one can also make use of (4) since, x_i and $\frac{\partial}{\partial x_i}$ serve as the creation and annihilation operators respectively. The ground-state can be chosen: $\frac{\partial}{\partial x_i}\phi_0 = 0$, for $i = 1, 2, \cdots, N$. The excited states are given by the monomials $\prod_l^N x_i^{n_l}$ taken in a symmetric form, with respect to the exchange of particle coordinates; here, $n_l = 0, 1, 2, \cdots$ and $E = \sum_l n_l + E_0$. There exist several related basis sets for these functions, invariant under the action of the symmetric group S_N, viz., Schur functions, monomial and complete symmetric functions [17]. The eigenstates of CS system spanning the n-th energy level can be written as

$$\Psi_n = \psi_0[\exp\{-\hat{A}/2\} \ S_n] = \psi_0 \ P_n \qquad . \tag{7}$$

Here, $P_n \equiv \exp\{-\hat{A}/2\}S_n$ are totally symmetric, inhomogeneous polynomials and S_ns are any of the above mentioned symmetric polynomials. One special choice of S_n's is provided by the Jack polynomials [17]. This basis set have been recently studied by T.H. Baker and P.J. Forrester, who also obtain the ground state normalization [15].

Although the Hamiltonian consists of decoupled oscillators, it is of deep interest to note that, the individual particle states created by the action of the powers of a single a_i^+ on the ground state give rise to functions which contain negative powers of the particle coordinates and hence is in general, not normalizable. But, the appropriate symmetric combinations of these non-normalizable functions results in polynomials, which are normalizable with respect to the ground state wavefunction as the weight function. This fact reveals the many-particle

correlation inherent in the CS model and allows only the N-particle states to belong to the physical Hilbert space.

The quantum integrablility of the CS model has been proved earlier by identifying an infinite number of constants of motion [13]. It is however clear that one should look for *only* N linearly independent conserved quantities, since this is a N particle system in one dimension. From (6), one can check that $[H\ ,\ H_k] = [H_i\ ,\ H_j] = 0; i, j, k = 1, 2, \cdots, N$. Therefore, the set $\{H_1, H_2, \cdots, H_N\}$ provides the N such conserved quantities. One can construct, N linearly independent symmetric conserved quantities from the elementary symmetric polynomials, since they form a complete set:

$$I_1 = \sum_{1\leq i\leq N} H_i\ , I_2 = \sum_{1\leq i<j\leq N} H_i H_j\ , I_3 = \sum_{1\leq i<j<k\leq N} H_i H_j H_k, \cdots, I_N = \prod_{i=1}^{N} H_i. \tag{8}$$

Here, we would like to point out that, one can also construct the following type of conserved quantities using H_i and a_i^+:

$$I = \sum_i H_i^2 + \alpha \sum_{\substack{i,j \\ i\neq j}} \frac{a_i^+ + a_j^+}{a_i^+ - a_j^+}(H_i - H_j) \tag{9}$$

and its eigenfunctions are $Z^\alpha G(\exp\{-A/2\} J_\lambda)$ [15], which are also the eigenfunctions of the CS Hamiltonian; here, J_λ is a Jack polynomial of degree λ.

Akin to the free oscillator case, one can define a linear $W_{1+\infty}$ algebra for the CS model, using a_i^- and a_i^+. We choose one of the basis for the generators of the $W_{1+\infty}$ as $L_{m,n} = \sum_{i=1}^N (a_i^+)^{m+1} (a_i^-)^{n+1}$, for $m, n \geq -1$ [18]. These will obey the linear relation

$$[L_{m,n}\ ,\ L_{r,s}] = \sum_{p=0}^{\mathrm{Min}(n,r)} \frac{(n+1)!(r+1)!}{(n-p)!(r-p)!(p+1)!} L_{m+r-p,n+s-p} - (n \leftrightarrow s, m \leftrightarrow r) \quad . \tag{10}$$

The highest weight vector obtained from $L_{m,n}\psi_0 = 0$ for $n > m \geq -1$ is nothing but the CS model ground-state wave function. One can also choose $W_n^{(s)} \equiv L_{n+s-2,s-2}$ for $s \geq 1$ and $n \geq 1 - s$; where, $W_n^{(s)}$ is the n-th Fourier mode of a spin s field. This result may find application in fractional quantum Hall effect, since the Laughlin type wave functions can be related to the CS model [5]. The previously known basis in the literature gave a non-linear realization of the above algebra, for which the coupling and $\hbar$ dependent non-linear terms

were not known explicitly [13]. We would like to remark that the above analysis can also be performed for the translationally invariant model [1].

In the following, we give an example of an interacting many-body Hamiltonian which can be solved in a manner analogous to the CS model.

We consider a recently studied two-dimensional model [19]:

$$H = -\frac{1}{2}\sum_i \vec{\nabla}_i^2 + \frac{1}{2}\sum_i \vec{r}_i^2 + \frac{g_1}{2}\sum_{\substack{i,j \\ i\neq j}} \frac{\vec{r}_i^2}{Y_{ij}^2} + \frac{g_2}{2}\sum_{\substack{i,j,k \\ i\neq j\neq k}} \frac{\vec{r}_j}{Y_{ij}}\frac{\vec{r}_k}{Y_{ik}} \qquad . \tag{11}$$

Here, $\vec{r}_i^2 = x_i^2 + y_i^2$ and $Y_{ij} = x_i y_j - x_j y_i$. The operator which brings the above Hamiltonian to the diagonal form *i.e.*, $\hat{U}^{-1}\ H\ \hat{U} = \sum_i x_i \frac{\partial}{\partial x_i} + \sum_i y_i \frac{\partial}{\partial y_i} + E_0$, is given by

$$\hat{U} = \prod_{i<j}^{N} |Y_{ij}|^g\ \exp\{-\frac{1}{2}\sum_i \vec{r}_i^2\}\ \exp\{-\hat{C}/2\} \qquad , \tag{12}$$

where, $\hat{C} \equiv \frac{1}{2}\sum_i(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2}) - g\sum_{\substack{i,j \\ i\neq j}} \frac{1}{Y_{ij}}(x_j\frac{\partial}{\partial y_i} - y_j\frac{\partial}{\partial x_i})$, $E_0 = N + gN(N-1)$, $g_1 = g(g-1)$ and $g_2 = g^2$. One set of excited states and the corresponding energy spectrum are given by $\Psi_{n,l=0} = \hat{U} S_n(r_1^2, r_2^2, \cdots, r_N^2)$ and $E_{n,l=0} = 2n + E_0 = 2\sum_{k=1}^N n_k + E_0$ respectively; here, l refers to the angular momentum quantum number and the symmetric function $S_n(r_1^2, r_2^2, \cdots, r_N^2)$ can be constructed from any of the earlier mentioned basis. Another set of eigenstates for $l \neq 0$ can be constructed by choosing the symmetric function to be $S_{n_1,n_2} = (\sum_i z_i^2)^{n_1}(\sum_i \bar{z}_i^2)^{n_2}$; here, z_i $(\bar{z}_i) = x_i + iy_i$ $(x_i - iy_i)$ and the corresponding energy eigenvalues are $E_{n_1,n_2} = n_1 + n_2 + E_0$.

As is clear from the above analyses, the D dimensional N particle Hamiltonians which can be brought through a suitable transformation to the generalized form: $\tilde{H} = \sum_{l=1}^D \sum_{i=1}^N x_i^{(l)} \frac{\partial}{\partial x_i^{(l)}} + c + \hat{F}$ can also be mapped to $\sum_{l=1}^D \sum_{i=1}^N x_i^{(l)} \frac{\partial}{\partial x_i^{(l)}} + c$ by $\exp\{-d^{-1}\hat{F}\}$; where, the operator $\hat{F}$ is any homogeneous function of $\frac{\partial}{\partial x_i^{(l)}}$ and $x_i^{(l)}$ with degree d and c is a constant. For the normalizability of the wave functions, one needs to check that the action of $\exp\{-d^{-1}\hat{F}\}$ on an appropriate linear combination of the eigenstates of $\sum_{l=1}^D \sum_{i=1}^N x_i^{(l)} \frac{\partial}{\partial x_i^{(l)}}$ yields a polynomial solution. These Hamiltonians fall in one class in the sense that, any member of the class can be mapped to the other via free harmonic oscillators. This result may find application in other branches of physics and mathematics.

Since, the mapping between the CS model and harmonic oscillators was established by a similarity transformation, one can immediately construct the coherent state for CS model [20] starting from the coherent state of harmonic oscillators. Akin to the harmonic oscillator coherent states, these will be extremely useful for finding the classical limit of the quantum CS model [21]. We also remark that, the presence of the $W_{1+\infty}$ algebra in the oscillator representation indicates the possibility of having Kadomtsev-Petviashvili (KP) type non-linear equations in this model [22]. Work along the above lines as well as the application of our technique to spin Calogero models are currently under progress and will be reported elsewhere.

In conclusion, we have shown the equivalence of the CS model to N free harmonic oscillators; this led to a straightforward construction of the eigenstates, the N conserved quantities responsible for the quantum integrability of the system and the generators of the *linear* $W_{1+\infty}$ algebra. CS type many-body Hamiltonian in higher-dimension was also solved in an analogous manner.

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